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Quantum group symmetry of the quantum Hall effect on non-flat surfaces

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Abstract. After showing that the magnetic translation operators are not the symmetries of the quantum Hall effect (QHE) on non-flat surfaces, we show that another set of operators which leads to the quantum group symmetries for some of these surfaces exists. As a first example we show that the $su(2)$ symmetry of the QHE on a sphere leads to $su_q(2)$ algebra in the equator. We explain this result by a contraction of $su(2)$. Second, with the help of the symmetry operators of QHE on the Poincaré upper half plane, we will show that the ground-state wavefunctions form a representation of the $su_q(2)$ algebra.

1. Introduction

After the discovery of the quantum Hall effect (QHE) [1] and the fractional quantum Hall effect (FQHE) [2], Laughlin [3] introduced his interacting electron model and showed that the incompressible quantum fluid can explain the appearance of the plateaux in the FQHE for the filling factor $\nu = 1/m$, where m is an odd integer. In recent years there have been many attempts to explain this feature of incompressibility by the symmetries of the quantum mechanics of the two-dimensional planar motion of a non-relativistic particle in a uniform magnetic field. Recently Kogan [4] and Sato [5], by using the magnetic translation operator, showed that there exists a quantum group symmetry in this problem. They found that the following combination of the magnetic translation operator, $T_{\boldsymbol{a}} = \exp(\boldsymbol{a} \cdot (\boldsymbol{\nabla} + \mathrm{i}\boldsymbol{A}))$, where $\boldsymbol{a}$ is a constant vector and $\boldsymbol{A}$ is the electromagnetic potential, could represent the $su_q(2)$ algebra:

$$J_{\pm} = \frac{1}{q - q^{-1}}(\alpha_{\pm} T_{\pm\boldsymbol{a}} + \beta_{\pm} T_{\pm\boldsymbol{b}}) \qquad q^{2J_3} = T_{\boldsymbol{b}-\boldsymbol{a}} \tag{1}$$

where $q = \exp(\frac{1}{2}\mathrm{i}\boldsymbol{B} \cdot (\boldsymbol{a} \times \boldsymbol{b}))$ and $\alpha_+\beta_- = \beta_+\alpha_- = -1$. We recall in passing that the $su_q(2)$ algebra is the q-deformation of the universal enveloping algebra of the Lie algebra $su(2)$, given by

$$[J_3, J_{\pm}] = \pm J_{\pm} \qquad [J_+, J_-] = \frac{q^{2J_3} - q^{-2J_3}}{q - q^{-1}}. \tag{2}$$

Let $W_{n,\bar{n}} = T_{\boldsymbol{a}}$ where $a_i = \epsilon_{ij} n_j$, then it can be shown that $W_{n,\bar{n}}$ satisfies the Fairlie–Fletcher–Zachos (FFZ) trigonometric algebra [6]. This algebra in the weak field limit ($B \to 0$) leads to the w_∞ algebra, the algebra of the area-preserving diffeomorphism [7].

The simultaneous presence of the area-preserving diffeomorphism (the $su_q(2)$ symmetry) and incompressibility suggest that there may be a connection between them†. The same symmetry was also found for the topological torus [8].

In this paper we will study the same quantum group symmetry for non-flat surfaces. In section 2 we will show that the magnetic translation operators are the symmetries of the Hamiltonian only when the metric is flat. Therefore in the case of non-flat surfaces we must first look for the symmetry operators and then try to find any quantum group stucture of them. In section 3 we will begin with the first non-trivial surface, the sphere. The problem of the motion of the electrons in the presence of a magnetic monopole and when the electrons are restricted to move on a sphere was first considered by Haldane [9]. He formulated the problem such that the symmetry algebra of the Hamiltonian is the $su(2)$ algebra, with generators which are represented by a special combination of the rotation and gauge transformation operators. We will consider the group elements of this algebra with non-constant parameters, that is the set of maps from S^2 to $SU(2)$. By studying its multiplication law we will recover the FFZ algebra for a special region of the sphere. We will try to explain this appearance of $su_q(2)$ from $su(2)$ by a special contraction of $su(2)$.

It is well known that the group of automorphisms of any genus $g \geqslant 2$ compact Riemann surface is discrete. So to look for any quantum group symmetry we have to consider non-compact surfaces. Here we consider the Poincaré upper half plane in section 3, and we will show that the $su_q(2)$ algebra is the symmetry of this surface.

When this paper was nearly finished, we became aware of a preprint [10] in which the quantum group symmetry of a system of electrons on a sphere had been discussed.

2. Symmetry properties of the magnetic translation operator

Consider a particle on a Riemann surface interacting with a monopole field, that is the integral of the field strength out of the surface differs from zero. The natural definition of the constant magnetic field is [11]

$$F_{\mu\nu} = B\sqrt{g}\epsilon_{\mu\nu} \tag{3}$$

and the Hamiltonian of the electron is given by

$$H = \frac{1}{2m}\frac{1}{\sqrt{g}}(\partial_\mu - \mathrm{i}A_\mu)\sqrt{g}g^{\mu\nu}(\partial_\nu - \mathrm{i}A_\nu) = \frac{1}{2m}\boldsymbol{\nabla}^2 + \frac{B}{2m} \tag{4}$$

where $\nabla_\mu = \partial_\mu - \mathrm{i}A_\mu$. In [11] this Hamiltonian was solved by choosing some special metrics.

Now consider the magnetic translation operator $T_{\boldsymbol{\xi}} = \mathrm{e}^{\xi^\mu D_\mu}$ which acts on scalars. Here $\boldsymbol{\xi} = \xi^\mu\partial_\mu$ is a vector field and $D_\mu = \partial_\mu + \mathrm{i}A_\mu$. It can be shown that the operators $T_{\boldsymbol{\xi}}$ is a symmetry operator only when

$$[D_\mu, \nabla_\nu] = -\partial_\mu A_\nu - \partial_\nu A_\mu = 0. \tag{5}$$

In other words we must be able to choose the symmetric gauge. But solving equations (4) and (5) gives the following condition

$$\partial_\mu(B\sqrt{g}) = 0. \tag{6}$$

As B is a constant, this equation shows that the symmetric gauge is possible only when the surface is flat. Therefore $T_{\boldsymbol{\xi}}$ does not commute with the Hamiltonian when the surface is not flat.

† We would like to thank one of the referees for his comment on this point. The incompressibility feature of the FQHE is a collective behaviour of the electrons results from the interacting Hamiltonian. Therefore the invariance of the single-particle problem under the area-preserving diffeomorphisms by itself does not imply incompressibility.

3. Electron on a sphere

Consider an electron which is restricted to move on a sphere with radius R, in the presence of a magnetic monopole at the centre of the sphere. The flux of the magnetic field B is quantized by the Dirac quantization condition $B = \hbar S/eR^2$, where S is half integer. The single-particle Hamiltonian is [9]

$$H = \frac{\mathbf{\Lambda}^2}{2mR^2} \tag{7}$$

where $\mathbf{\Lambda} = \boldsymbol{r} \times [-\mathrm{i}h\boldsymbol{\nabla} + e\boldsymbol{A}]$, $\boldsymbol{A}$ satisfies $\boldsymbol{\nabla} \times \boldsymbol{A} = B\boldsymbol{\Omega}$ ($\boldsymbol{\Omega} = R/R$) and $\mathbf{\Lambda} \cdot \boldsymbol{\Omega} = 0$. By using gauge freedom, the electromagnetic potential can be taken as

$$\boldsymbol{A} = -\frac{\hbar S}{eR} \cot\theta\hat{\varphi}. \tag{8}$$

The eigenvalues of $\mathbf{\Lambda}^2$ are $(\ell(\ell+1) - S^2)\hbar^2$ and the first Landau level is obtained for $\ell = s$ and is equal to $\hbar w_{\mathrm{c}}/2$ ($w_{\mathrm{c}} = eB/m$). It can be shown that the Λ_i's satisfy the following relations:

$$[\Lambda_i, \Lambda_j] = \mathrm{i}\hbar\epsilon_{ijk}(\Lambda_k - \hbar S\Omega_k) \tag{9}$$

and $\boldsymbol{L} = \mathbf{\Lambda} + \hbar s\boldsymbol{\Omega}$ generates the $su(2)$ algebra

$$[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_k. \tag{10}$$

The operators L_i are the generators of the symmetries of the Hamiltonian: $[H, L_i] = 0$.

Now consider the group elements of this algebra

$$R_\xi = \exp\left(\frac{\mathrm{i}}{\hbar}\boldsymbol{L} \cdot \frac{\boldsymbol{\xi}}{R}\right) \qquad \text{and} \qquad R_\eta = \exp\left(\frac{\mathrm{i}}{\hbar}\boldsymbol{L} \cdot \frac{\boldsymbol{\eta}}{R}\right) \tag{11}$$

where $\boldsymbol{\xi} = \xi\hat{\theta}$ and $\boldsymbol{\eta} = \eta\hat{\varphi}$. The product of these operators is

$$R_\xi R_\eta = \exp\left\{\frac{\mathrm{i}}{\hbar R}(\boldsymbol{\xi} + \boldsymbol{\eta}) \cdot \mathbf{\Lambda} - \frac{1}{2R^2\hbar^2}[\boldsymbol{\xi} \cdot \mathbf{\Lambda}, \boldsymbol{\eta} \cdot \mathbf{\Lambda}] + \frac{\mathrm{i}}{3\hbar R}[\boldsymbol{\xi} \cdot \mathbf{\Lambda}, M] + \cdots\right\} \tag{12}$$

where M denotes the second term of the exponent. A simple calculation shows that

$$[\boldsymbol{\xi} \cdot \mathbf{\Lambda}, \boldsymbol{\eta} \cdot \mathbf{\Lambda}] = -\mathrm{i}\hbar\xi\eta(\hbar S + \cot\theta\Lambda_\theta) \tag{13}$$

where

$$\Lambda_\theta = \hbar\left(S\cot\theta + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right).$$

Now the equality $\boldsymbol{\xi} \cdot \mathbf{\Lambda} = \xi\Lambda_\theta$ implies that $[\boldsymbol{\xi} \cdot \mathbf{\Lambda}, M] = 0$. Therefore

$$R_\xi R_\eta = \exp\left\{\frac{\mathrm{i}}{\hbar R}(\boldsymbol{\xi} + \boldsymbol{\eta}) \cdot \mathbf{\Lambda} + \frac{\mathrm{i}\xi\eta}{2R^2\sin^2\theta}\left(S + \cos\theta\frac{\partial}{\partial\varphi}\right)\right\}. \tag{14}$$

If we restrict ourselves to the region $\theta = \frac{\pi}{2}$, we find that

$$R_\xi R_\eta = \exp\left(\frac{\mathrm{i}eB\xi\eta}{2\hbar}\right)R_{\xi+\eta}. \tag{15}$$

Comparing this equation with the relation satisfied by the magnetic translation operator T_ξ [4]; i.e.

$$T_\xi T_\eta = \exp\left(\frac{\mathrm{i}e}{2\hbar}\boldsymbol{B} \cdot (\boldsymbol{\xi} \times \boldsymbol{\eta})\right)T_{\xi+\eta} \tag{16}$$

we see that the algebra of the operator valued map $R_{\boldsymbol{\xi}} : S^2 \to SU(2)$, when restricted to the circle $\theta = \pi/2$, is isomorphic to the magnetic translation algebra. Therefore by using the same construction as in equation (1), one can also derive the $su_q(2)$ algebra out of the operator $R_{\boldsymbol{\xi}}$ with

$$q = \exp\left(\frac{\mathrm{i}e}{2\hbar}\boldsymbol{B}\cdot(\boldsymbol{\xi}\times\boldsymbol{\eta})\right).$$

This result may be made plausible if we note that the magnetic translation operator $T_{\boldsymbol{\xi}}$ can also be expressed as [4]

$$T_{\boldsymbol{\xi}} = W_{n,\bar{n}} = \exp(\tfrac{1}{2}(nb^{\dagger} - \bar{n}b)) \tag{17}$$

where $b = 2p_{\bar{z}} - \frac{1}{2}\mathrm{i}Bz$ and $b^{\dagger} = 2p_z + \frac{1}{2}\mathrm{i}B\bar{z}$ and they satisfy the Heisenberg algebra: $[b, b^{\dagger}] = 2B$. On the other hand the $su(2)$ algebra which is used in the construction of $R_{\boldsymbol{\xi}}$ operators can be contracted to the Heisenberg algebra plus a rotation as follows. Let $H, X^{\pm}$ be the generators of $su(2)$:

$$[H, X^{\pm}] = \pm 2X^{\pm} \qquad [X^+, X^-] = H. \tag{18}$$

Define new generators H' and $P^{\pm}$ as in $H = H' + 1/\epsilon^2$ and $P^{\pm} = \epsilon X^{\pm}$, then

$$[H', P^{\pm}] = \pm 2P^{\pm} \qquad [P^+, P^-] = \epsilon^2 H' + 1. \tag{19}$$

At the limit $\epsilon \to 0$ we have

$$[H', P^{\pm}] = \pm 2P^{\pm} \qquad [P^+, P^-] = 1. \tag{20}$$

Therefore it is reasonable to expect that the algebra of the magnetic translation operator $T_{\boldsymbol{\xi}}$ which is the exponential of only the Heisenberg algebra , is obtained from the complicated algebra (12), when we restrict ourselves to the $\theta = \pi/2$ circle.

4. Electron on the Poincaré upper half plane

In this section we consider the Poincaré upper half plane $H = \{z = x + \mathrm{i}y, y > 0\}$, with the following metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}. \tag{21}$$

For a covariantly constant magnetic field B, a particular gauge choice leads to

$$A_z = A_{\bar{z}} = \frac{B}{2y}. \tag{22}$$

In this gauge the Hamiltonian (4) reduces to (for simplicity we take $m = 2$)

$$H = -y^2\partial\bar{\partial} + \frac{\mathrm{i}B}{2}y(\partial + \bar{\partial}) + \frac{B^2}{4} \tag{23}$$

and the ground states with energy $B/4$ are given by the solutions of the equation

$$\bar{\nabla}\psi_0 = \left(\bar{\partial} + \frac{B}{2\mathrm{i}y}\right)\psi_0 = 0 \tag{24}$$

which are $\psi_0(z, \bar{z}) = y^B\psi_0(z)$.

It can be easily checked that there are two operators which commute with the Hamiltonian (23):

$$L_1 = \partial + \bar{\partial} = \partial_x \qquad L_2 = z\partial + \bar{z}\bar{\partial} = x\partial_x + y\partial_y. \tag{25}$$

L_1 and L_2 are the generators of a subalgebra of $sl(2, R)$. Now let $b = L_1$ and $b^\dagger = L_1^{-1} L_2$ where $[b, b^\dagger] = 1$. Then we can choose the ground-state wavefunctions to be the eigenfunctions of $b^\dagger$. By a direct calculation it can be shown that

$$b^\dagger \psi_0(\lambda|z, \bar{z}) = \lambda \psi_0(\lambda|z, \bar{z}) \tag{26}$$

where

$$\psi_0(\lambda|z, \bar{z}) = y^B (\lambda - z)^{-B}. \tag{27}$$

Then if we consider the symmetry operator $T_{\boldsymbol{\xi}} = \exp(\xi_1 b + \xi_2 b^\dagger)$ we obtain

$$T_{\boldsymbol{\xi}} \psi_0(\lambda|z, \bar{z}) = \exp(\xi_2 \lambda - \tfrac{1}{2}\xi_1 \xi_2) \psi_0(\lambda - \xi_1|z, \bar{z}). \tag{28}$$

Now it can be verified that the generators of $su_q(2)$ are

$$J_+ = \frac{T_{\boldsymbol{\xi}} - T_{\boldsymbol{\eta}}}{q - q^{-1}} \qquad J_- = \frac{T_{-\boldsymbol{\xi}} - T_{-\boldsymbol{\eta}}}{q - q^{-1}}$$
$$q^{2J_0} = T_{\boldsymbol{\xi} - \boldsymbol{\eta}} \qquad \text{where } q = \exp(\tfrac{1}{2}\boldsymbol{\xi} \times \boldsymbol{\eta}) \tag{29}$$

and the ground-state wavefunctions are a representation of this algebra

$$J_+ \psi_0(\lambda|z, \bar{z}) = [1/2 - \lambda/\xi_1]_q \psi_0(\lambda - \xi_1|z, \bar{z})$$
$$J_- \psi_0(\lambda|z, \bar{z}) = [1/2 + \lambda/\xi_1]_q \psi_0(\lambda + \xi_1|z, \bar{z})$$
$$q^{\pm J_0} \psi_0(\lambda|z, \bar{z}) = q^{\mp \lambda/\xi_1} \psi_0(\lambda|z, \bar{z}) \tag{30}$$

where the quantum symbol $[x]_q$ is defined by

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{31}$$

In this way we have shown the quantum group symmetry of QHE on the Poincaré upper half plane. Note that we only considered single-particle wavefunctions. The case of an interacting system with Laughlin wavefunctions will be studied elsewhere.

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References

[1] Klitzing K V, Dorda G and Pepper M 1980 *Phys. Rev. Lett.* **45** 494
[2] Tsui D C, Stromer H L and Gossard A C 1982 *Phys. Rev. Lett.* **48** 1559
[3] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1395
[4] Kogan I I 1994 *Int. J. Mod. Phys.* A **9** 3887
[5] Sato H T 1994 *Mod. Phys. Lett.* **9A** 451
[6] Fairlie D B, Fletcher P and Zachos C K 1989 *Phys. Lett.* **218B** 203; 1990 *J. Math. Phys.* **31** 1088
Fairlie D B and Zachos C K 1989 *Phys. Lett.* **224B** 101
[7] For instance see: Shen X 1992 *Int. J. Mod. Phys.* A **7** 6953
[8] Sato H T 1994 *Mod. Phys. Lett.* **9A** 1819
[9] Haldane F D 1983 *Phys. Rev. Lett.* **51** 605
[10] Aizawa N, Sachse S and Sato H T 1995 Laughlin states on the sphere as representation of $U_q(sl(2))$ hep-th/9412017
[11] Comtet A 1987 *Ann. Phys.* **173** 185
Antoine M, Comtet A and Ouvry S 1990 *J. Phys. A: Math. Gen.* **23** 3699
Bolte J and Steiner F 1991 *J. Phys. A: Math. Gen.* **24** 3817
[12] Iengo R and Li D 1994 *Nucl. Phys.* B **413** 735